

CONSTELLATIONS OF GAPS IN ERATOSTHENES SIEVE

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ABSTRACT. A few years ago we identified a recursion that works directly with the gaps among the generators in each stage of Eratosthenes sieve. This recursion provides explicit enumerations of sequences of gaps among the generators, which sequences are known as constellations.

Over the last year we identified a discrete linear system that exactly models the population of any gap across all stages of the sieve. In August 2014 we summarized our results from analyzing this discrete model on populations of single gaps. This paper extends the discrete system to model the populations of constellations of gaps.

The most remarkable result is a *strong Polignac result on arithmetic progressions*. We had previously established that the equivalent of Polignac's conjecture holds for Eratosthenes sieve – that every even number arises as a gap in the sieve, and its population converges toward the ratio implied by Hardy and Littlewood's Conjecture B. Extending that work to constellations, we here establish that for any even gap g , if p is the maximum prime such that $p^\# | g$ and P is the next prime larger than p , then for every $2 \leq j_1 < P - 1$, the constellation g, g, \dots, g of length j_1 arises in Eratosthenes sieve. This constellation corresponds to an arithmetic progression of $j_1 + 1$ consecutive candidate primes.

1. INTRODUCTION

This paper is a technical note that builds on our previous paper, *Eratosthenes sieve and the gaps between primes* [8]. As such, we will assume familiarity with the notations, models, methods, and results of that work.

A *constellation among primes* [13] is a sequence of consecutive gaps between prime numbers. Let $s = a_1 a_2 \cdots a_k$ be a sequence of k numbers. Then s is a constellation among primes if there exists a sequence of $k + 1$ consecutive prime numbers $p_i p_{i+1} \cdots p_{i+k}$ such that for each $j = 1, \dots, k$, we have the gap $p_{i+j} - p_{i+j-1} = a_j$.

We do not study the gaps between primes directly. Instead, we study the cycle of gaps $\mathcal{G}(p^\#)$ at each stage of Eratosthenes sieve. Here, $p^\#$ is the *primorial* of p , which is the product of all primes from 2 up to and

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including p . $\mathcal{G}(p^\#)$ is the cycle of gaps among the generators of $\mathbb{Z} \bmod p^\#$. These generators and their images through the counting numbers are the candidate primes after Eratosthenes sieve has run through the stages from 2 to p . All of the remaining primes are among these candidates, but many of the candidates do not survive further stages of the sieve to be confirmed as primes.

At times we will speak of constellations *corresponding* to gaps between primes. We do this for intuitive convenience, to motivate why the particular constellations we study may be of interest. The results here are for populations within Eratosthenes sieve, and the constellations correspond to primes only to the extent that they survive the sieve.

In [8] we identified a discrete dynamic system that exactly models the populations of gaps from the cycle of gaps $\mathcal{G}(p_k^\#)$ in one stage of Eratosthenes sieve to the next, $\mathcal{G}(p_{k+1}^\#)$. We use the eigenstructure of this system to obtain asymptotic ratios between the populations of various gaps in the sieve.

With the following theorem, we generalize the discrete dynamic system for gaps to model the populations of constellations across stages of Eratosthenes sieve.

Theorem 1.1. *Let s be a constellation of length j_1 and of sum $|s|$. Let p_0 be a prime such that $|s| < 2p_1$.*

Let $n_j(s, p^\#)$ be the population of driving terms of length j for s in $\mathcal{G}(p^\#)$, with $j_1 \leq j \leq J$ and $p \geq p_0$. Then

$$\begin{aligned} (1) \quad n_s(p_k^\#) &= M_{j_1:J}|_{p_k} \cdot n_s(p_{k-1}^\#) \\ &= R \cdot \Lambda_{j_1:J}|_{p_k} \cdot L \cdot n_s(p_{k-1}^\#). \end{aligned}$$

in which L is the upper triangular Pascal matrix, R is the upper triangular Pascal matrix with alternating signs $(-1)^{i+j}$, and Λ is the diagonal matrix of eigenvalues $(p_k - j - 1)$ for $j_1 \leq j \leq J$.

As with our work on gaps, this explicit enumeration has two limitations. The first limitation is that all of the populations grow super exponentially by factors of $(p - j_1 - 1)$. To address this, we normalize the dynamic system by dividing by factors of $(p - j_1 - 1)$, which makes the dominant eigenvalue equal to 1.

The second limitation of Theorem 1.1 is the condition $|s| < 2p_1$. Because of this condition, to exactly model the population of a constellation we have to calculate its initial conditions in $\mathcal{G}(p^\#)$, for p very close to $|s|$. But $\mathcal{G}(p^\#)$ consists of $\phi(p^\#)$ gaps, which becomes unwieldy around $p = 37$:

p	length of $\mathcal{G}(p^\#)$
31	3.06561 <i>E10</i>
37	1.10362 <i>E12</i>
41	4.41448 <i>E13</i>
43	1.85408 <i>E15</i>

The eigenstructure of the dynamic system in Equation (1) enables us to extract the asymptotic behavior of the population for constellations, if we can derive the initial conditions for a cycle of gaps $\mathcal{G}(p^\#)$ under much weaker conditions than those of Theorem 1.1.

Extending the analogue of Polignac's conjecture for Eratosthenes sieve [8], we are able to establish a strong Polignac result for arithmetic progressions: that for any $g = 2n$, every feasible repetition $s = g, \dots, g$ occurs in Eratosthenes sieve. To the extent that these repetitions survive the sieve, they correspond to consecutive primes in arithmetic progression with gap g .

2. MODELING THE POPULATIONS OF CONSTELLATIONS

2.1. Summarizing the model for gaps. In [7] we developed a dynamic system that exactly models the populations of gaps through the stages of Eratosthenes sieve. We restate the model briefly here.

Let g be a gap (a constellation of length 1) with driving terms up to length J . Let p_0 be a prime such that $g < 2p_1$. Then

$$\begin{aligned}
 n_g(p_k^\#) &= M|_{p_k} \cdot n_g(p_{k-1}^\#) \\
 &= \begin{bmatrix} p_k - 2 & 1 & 0 & \cdots & 0 \\ 0 & p_k - 3 & 2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & & & J - 1 \\ 0 & \cdots & & & p_k - J - 1 \end{bmatrix} \cdot n_g(p_{k-1}^\#) \\
 &= R \cdot \Lambda(p_k) \cdot L \cdot n_g(p_{k-1}^\#)
 \end{aligned}$$

with

$$\Lambda(p_k) = \text{diag}[p_k - 2, p_k - 3, \dots, p_k - J - 1].$$

Since the eigenvectors do not depend on the prime p , the dynamic system can be expressed in terms of the initial conditions.

$$\begin{aligned}
 (2) \quad n_g(p_k^\#) &= M|_{p_k} \cdots M|_{p_1} \cdot n_g(p_0^\#) \\
 &= R \cdot \Lambda^k \cdot L \cdot n_g(p_0^\#)
 \end{aligned}$$

with

$$\Lambda^k = \text{diag} \left[\prod_1^k (p_i - 2), \prod_1^k (p_i - 3), \dots, \prod_1^k (p_i - J - 1) \right].$$

Dividing by $p - 2$ at each iteration, we obtain the normalized population model for gaps

$$(3) \quad \begin{aligned} w_g(p_k^\#) &= M|_{p_k} \cdots M|_{p_1} \cdot w_g(p_0^\#) \\ &= R \cdot \Lambda^k \cdot L \cdot w_g(p_0^\#) \end{aligned}$$

and here

$$\Lambda^k = \text{diag} \left[1, \prod_1^k \frac{p_i - 3}{p_i - 2}, \prod_1^k \frac{p_i - 4}{p_i - 2}, \dots, \prod_1^k \frac{p_i - J - 1}{p_i - 2} \right].$$

The matrix L is an upper triangular Pascal matrix, whose entries are the binomial coefficients, and R is an upper triangular matrix whose entries have alternating signs.

$$R_{ij} = \begin{cases} (-1)^{i+j} \binom{j-1}{i-1} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases} \quad \text{and} \quad L_{ij} = \begin{cases} \binom{j-1}{i-1} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$$

This dynamic system encompasses constellations as driving terms for gaps. To adapt the system to focus directly on the population of a constellation and its driving terms, we only have to shift the eigenvalues.

2.2. Extending the model to constellations. The model for gaps relies on keeping track of the internal closures and external closures for the driving terms for gaps. For constellations, we need a richer concept than external closures, and so we define the *boundary closures* for a driving term for a constellation.

Let s be a constellation of length j_1 , $s = g_1, g_2, \dots, g_{j_1}$. A driving term \tilde{s} for s will have the form $\tilde{s} = \tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_{j_1}$ in which $|\tilde{s}_i| = g_i$ for each i .

We call the closures within an \tilde{s}_i the *interior closures* for \tilde{s} , and the exterior closures for each \tilde{s}_i are the *boundary closures* for \tilde{s} . Interior closures preserve the copy of \tilde{s} as a driving term for s but of shorter length. The $j_1 + 1$ boundary closures remove the copy of \tilde{s} from being a driving term for s .

We are now ready to prove Theorem 1.1.

Proof. of Theorem 1.1: Let s be a constellation of length j_1 and of sum $|s|$. Let p_0 be a prime such that $|s| < 2p_1$.

Let $n_{s,j}(p^\#)$ be the population of driving terms of length j for s in $\mathcal{G}(p^\#)$, with $j_1 \leq j \leq J$ and $p \geq p_0$.

Consider the recursion from $\mathcal{G}(p_{k-1}^\#)$ to $\mathcal{G}(p_k^\#)$, with $k \geq 1$. Let \tilde{s} be a particular occurrence of a driving term of length j in $\mathcal{G}(p_{k-1}^\#)$. Since

$|s| < 2p_k$, for the p_k copies of \tilde{s} initially created during step R2 of the recursion, the $j+1$ closures of step R3 all occur in different copies.

Of the p_k initial copies of \tilde{s} , the boundary closures eliminate j_1+1 copies as driving terms for s . The $j-j_1$ interior closures produce $j-j_1$ driving terms of length $j-1$, and p_k-j-1 copies of \tilde{s} survive intact as driving terms of length j .

We can express this in the dynamic system:

$$\begin{aligned} n_s(p_k^\#) &= M_{j_1:J}(p_k) \cdot n_s(p_{k-1}^\#) \\ &= \begin{bmatrix} p_k - j_1 - 1 & 1 & 0 & \cdots & 0 \\ 0 & p_k - j_1 - 2 & 2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & & & J - j_1 \\ 0 & \cdots & & & p_k - J - 1 \end{bmatrix} \cdot n_g(p_{k-1}^\#) \end{aligned}$$

Note that $M_{j_1:J}(p)$ is a $(J-j_1+1) \times (J-j_1+1)$ matrix.

The matrix $M_{j_1:J}(p)$ has the same eigenvectors as the matrix for gaps (of size $J-j_1+1$). The eigenstructure for $M_{j_1:J}(p)$ is given by

$$M_{j_1:J}(p) = R \cdot \Lambda_{j_1:J}(p) \cdot L$$

in which R is still the upper triangular Pascal matrix with alternating signs, and L is the upper triangular Pascal matrix. These do not depend on the prime p . The eigenvalues are

$$\Lambda_{j_1:J}(p) = \text{diag}[p-j_1-1, p-j_1-2, \dots, p-J-1].$$

Since the eigenvectors do not depend on the prime p , the dynamic system can easily be expressed in terms of the initial conditions.

$$\begin{aligned} (4) \quad n_s(p_k^\#) &= M_{j_1:J}|_{p_k} \cdots M|_{p_1} \cdot n_s(p_0^\#) \\ &= R \cdot \Lambda_{j_1:J}^k \cdot L \cdot n_s(p_0^\#) \end{aligned}$$

with

$$\Lambda_{j_1:J}^k = \text{diag} \left[\prod_1^k (p_i - j_1 - 1), \prod_1^k (p_i - j_1 - 2), \dots, \prod_1^k (p_i - J - 1) \right].$$

□

2.3. Normalizing the populations. In the large, the population of a constellation of length j_1 grows primarily by a factor of $(p-j_1-1)$. So all constellations of length j ultimately become more numerous than any constellation of length $j+1$, and comparing the asymptotic populations of constellations of different lengths is thereby trivial.

On the other hand, to determine the relative occurrence among constellations of a given length j_1 , we divide by the factor $(p-j_1-1)$. We define

the *normalized population* of a constellation s of length j_1 and driving terms up to length J as

$$w_s(p^\#) = \left(\prod_{q > j_1+1}^p \frac{1}{q - j_1 - 1} \right) \cdot n_s(p^\#) = \frac{1}{\phi_{j_1+1}(p^\#)} \cdot n_s(p^\#).$$

We introduce the functions $\phi_i(p^\#) = \prod_{q > i}^p (q - i)$.

Definition. Let $Q = q_1 q_2 \cdots q_m$ be a product of distinct primes, with $q_1 < q_2 < \cdots < q_m$. We define

$$\phi_i(Q) = \prod_{q_j > i} (q_j - i).$$

Note that $\phi_1 = \phi$, the Euler totient function, over the defined domain - products of distinct primes. For the primorials we have

$$\phi_i(p^\#) = \prod_{q > i}^p (q - i).$$

For the normalized populations, the dynamic system becomes

$$(5) \quad w_s(p_k^\#) = R \cdot \Lambda_{j_1:J}^k \cdot L \cdot w_s(p_0^\#)$$

with

$$\Lambda_{j_1:J}^k = \text{diag} \left[1, \prod_1^k \frac{p_i - j_1 - 2}{p_i - j_1 - 1}, \prod_1^k \frac{p_i - j_1 - 3}{p_i - j_1 - 1}, \dots, \prod_1^k \frac{p_i - J - 1}{p_i - j_1 - 1} \right].$$

For these normalized populations, the dominant eigenvalue is now 1.

For gaps, this normalization corresponds nicely to taking the ratio of the population of the gap under consideration to the population of the gap $g = 2$. For $j_1 = 2$, we can again interpret this normalization as a ratio, in this case to the populations of constellations $s = 24$ or $s = 42$. These constellations have no driving terms longer than j_1 , and $n_{24}(p^\#) = n_{42}(p^\#)$, so there is no ambiguity. This correlation between the normalization and ratios to constellations consisting of 2's and 4's begins to break down at $j_1 = 3$ and collapses completely at $j_1 = 6$. For $j_1 = 3$, we have two constellations to choose from, $s = 242$ or $s = 424$, and there is ambiguity because $n_{424}(p^\#) = 2 \cdot n_{242}(p^\#)$. For $j_1 = 4$ we have $s = 2424$ and $s = 4242$, and for $j_1 = 5$ we have $s = 42424$. However, by looking at $\mathcal{G}(5^\#)$ we see that there are no constellations consisting only of 2's and 4's for length $j_1 \geq 6$.

The normalization does not necessarily provide ratios of the populations of constellations of length j_1 to the population of a known constellation. Instead, it provides relative populations to a (perhaps hypothetical) constellation without driving terms other than the constellation itself and with population $\phi_{j_1+1}(p^\#)$.

For example, for $j_1 = 3$ we have $p_0 = 5$ and $p_0 - j_1 - 1 = 1$. The symmetric constellation $s = 242$ has $w_{242}^\infty = 1$; it has no additional driving terms, it is symmetric, but it occurs at the middle of the cycle $\mathcal{G}(5^\#)$. In contrast, the symmetric constellation $s = 424$ has $w_{424}^\infty = 2$, even though it too has no additional driving terms.

2.4. Relative occurrence in the large. Equation (5) applies under the same conditions as Theorem 1.1 – that s is a constellation of length j_1 and driving terms up to length J , and p_0 is a prime such that $|s| < 2p_1$.

If we can get a count $n_s(p_0^\#)$ for s and all of its driving terms in $\mathcal{G}(p_0^\#)$ with $|s| < 2p_1$, then we can apply the eigenstructure to obtain the asymptotic number of occurrences w_s^∞ of s relative to other constellations of length j_1 as $p_k \rightarrow \infty$:

$$\begin{aligned} w_s^\infty &= \lim_{p_k \rightarrow \infty} w_{s,j_1}(p_k^\#) \\ &= \lim_{p_k \rightarrow \infty} \frac{n_{s,j_1}(p_k^\#)}{\phi_{j_1+1}(p_k^\#)} \\ &= L_1 \cdot w_s(p_0^\#) \end{aligned}$$

For this asymptotic result, we can soften the requirement of needing to work with p_0 such that $|s| < 2p_1$. For the full dynamic system to apply, this condition guarantees that for each occurrence \tilde{s} of a driving term for s in $\mathcal{G}(p_{k-1}^\#)$, the closures in forming $\mathcal{G}(p_k^\#)$ occur in distinct copies of \tilde{s} .

Since $L_1 = [1, 1, \dots, 1]$, to calculate w_s^∞ we only need to know that the $j_1 + 1$ closures that remove a copy of \tilde{s} from being a driving term occur in distinct copies. Other closures may occur inside those copies as well, but we need to know that under the recursion $p - j_1 - 1$ copies of \tilde{s} survive as driving terms for s .

Theorem 2.1. *Let s be a constellation of j_1 gaps*

$$s = g_1, g_2, \dots, g_{j_1}.$$

Let p_0 be the highest prime that divides any of the intervals $g_i + \dots + g_j$ with $1 \leq i \leq j \leq j_1$. Then for any $p \geq p_0$,

$$w_s^\infty = L_1 \cdot w_s(p^\#).$$

Proof. Let \tilde{s} be a driving term for s . From $\mathcal{G}(p_0^\#)$ on, the closures that remove copies of \tilde{s} from being driving terms for s all occur in distinct copies. Thus for any subsequent p_k ,

$$\left| n_s(p_k^\#) \right| = (p_k - j_1 - 1) \left| n_s(p_{k-1}^\#) \right|.$$

We cannot be certain about the lengths of all of the copies that survive as driving terms, but we do know how the total population grows – by exactly the factor $p_k - j_1 - 1$. and so for all $p_k > p_0$,

$$w_s^\infty = L_1 \cdot w_s(p_k^\#).$$

□

3. SPECIFIC CONSTELLATIONS

We now apply the population model to a few specific constellations. Pursuing the twin prime conjecture, we study how constellations that correspond to twin primes persist in Eratosthenes sieve. One of these, $s = 242$, has been studied extensively before [11, 12, 10]; this constellation corresponds to prime quadruplets, two pairs of twin primes separated by a gap of 4.

The other family of constellations that we study here are those corresponding to consecutive primes in arithmetic progression. These constellations consist of the same gap repeated, such as $s = 66$ or $s = 666$.

3.1. Constellations related to twin primes. Twin primes correspond to a gap $g = 2$. Here we are studying how gaps $g = 2$ arise in Eratosthenes sieve, and we do not address how many of these might survive the sieve to become gaps between primes (in this case between twin primes).

The gap $g = 2$ arises in several interesting constellations. The first, $s = 242$, corresponds to prime quadruplets. Prime quadruplets are the densest occurrence of four primes in the large, two pairs of twin primes separated by a gap of 4. The constellation $s = 242$ has no additional driving terms, $j_1 = J = 3$. We could use $p_0 = 3$, for which $n_{242}(3^\#) = [1]$. We have

$$w_{242}^\infty = 1.$$

3.1.1. Constellation $s = 2, 10, 2$. The next constellation we consider is $s = 2, 10, 2$, which corresponds to two pairs of twin primes separated by a gap of 10. Here $j_1 = 3$ and $J = 4$, for the driving terms 2642 and 2462. Using $p_0 = 7$, we count $n_{2,10,2}(7^\#) = [2, 6]$ for two occurrences of $s = 2, 10, 2$ and three occurrences each of the driving terms 2642 and 2462.

$$w_{2,10,2}(7^\#) = \frac{1}{\phi_4(7^\#)} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2 \end{bmatrix}$$

and

$$w_{2,10,2}^\infty = 8/3$$

This means that as $p_k \rightarrow \infty$, the number of occurrences of $s = 2, 10, 2$ in the sieve approaches $8/3$ times the number of occurrences of the constellation 242. Remember that these weights w^∞ are relative only to other constellations of the same length j_1 .

3.1.2. *Constellation* $s = 2, 10, 2, 10, 2$. The constellation $s = 2, 10, 2, 10, 2$ corresponds to three pairs of twin primes with gaps of $g = 10$ separating them. This constellation also contains two overlapping copies of $2, 10, 2$, and two overlapping driving terms for the constellation $12, 12$ to which we will return when we look at arithmetic progressions.

For $s = 2, 10, 2, 10, 2$, we have $j_1 = 5$ and $J = 7$. Since $|s| = 26$, for initial conditions we have to use $\mathcal{G}(13^\#)$.

$$n_{2,10,2,10,2}(13^\#) = \begin{bmatrix} 52 \\ 44 \\ 48 \end{bmatrix}$$

$$w_{2,10,2,10,2}(13^\#) = \frac{1}{\phi_6(13^\#)} \begin{bmatrix} 52 \\ 44 \\ 48 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 52 \\ 44 \\ 48 \end{bmatrix}$$

and

$$w_{2,10,2,10,2}^\infty = 144/35.$$

So among constellations of length 5, the constellation $s = 2, 10, 2, 10, 2$ occurs with a relative frequency of $144/35$. For length $j_1 = 5$, we can use the constellation 42424 as a reference. The population model shows that in the large, the constellation $s = 2, 10, 2, 10, 2$ occurs over four times as frequently as the constellation 42424.

3.1.3. *Constellation* $s = 2, 10, 2, 10, 2, 4, 2, 10, 2, 10, 2$. Along this line of inquiry into constellations that contain several 2s, we observe that the following constellation occurs in $\mathcal{G}(13^\#)$:

$$s = 2, 10, 2, 10, 2, 4, 2, 10, 2, 10, 2.$$

This is two copies of $2, 10, 2, 10, 2$ separated by a gap of 4. This corresponds to six pairs of twin primes, or twelve primes total, occurring in an interval of $|s| = 56$.

For this constellation s , we have $j_1 = 11$ and $J = 13$. It would theoretically be possible for each of the 10's to be produced through closures, so there could be driving terms of length up to 15, but when we inspect $\mathcal{G}(13^\#)$, we find that there are two copies of s , ten driving terms of length 12, twelve driving terms of length 13, and no driving terms of length 14 or 15.

Since $|s| = 56$, to use Theorem 1.1 we would need to use $p_0 = 23$ to employ the full dynamic system. However, to obtain the asymptotic results of Theorem 2.1 we can use $p_0 = 13$. We calculate $w_s^\infty = 24$. This means that relative to a constellation of length 11 with one occurrence in $\mathcal{G}(13^\#)$ and no additional driving terms, as the sieve continues the constellation s will occur approximately 24 times as often.

s	$ s $	j_1	J	p_0	$n(p_0^\#)$	ω^∞
242	8	3	3	5	[1]	1
424	10	3	3	5	[2]	2
2, 10, 2	14	3	4	7	[2, 6]	8/3
42424	16	5	5	7	[1]	1
2, 10, 2, 10, 2	26	5	7	13	[52, 44, 48]	144/35
2, 10, 2, 10, 2, 4, 2, 10, 2, 10, 2	56	11	13	13	[2, 10, 12]	24
66	12	2	4	5	[0, 2, 2]	2
12, 12	24	2	6	11	[0, 2, 20, 48, 58]	2
666	18	3	5	7	[0, 4, 2]	2

TABLE 1. Table of initial conditions and parameters for a few representative constellations. The population of a constellation of length j_1 grows primarily by a factor of $p - j_1 - 1$.

3.2. Consecutive primes in arithmetic progression. A sequence of $j_1 + 1$ consecutive primes in arithmetic progression corresponds to a constellation of j_1 identical gaps g . By considering residues, we easily see that for a sequence of $j_1 + 1$ primes in arithmetic progression, g must be divisible by every prime $p \leq j_1 + 1$. So for three consecutive primes in arithmetic progression, the minimal constellation is $s = 66$. For four consecutive primes in arithmetic progression, the minimal constellation is $s = 666$, and then for an arithmetic progression of five consecutive primes the minimal constellation is $s = 30, 30, 30, 30$.

3.2.1. Constellation $s = 66$. Let's now calculate the population of the constellation $s = 66$. This constellation corresponds to three consecutive primes in arithmetic progression: $p, p + 6, p + 12$. Since $|s| = 12$, we can still use $p_0 = 5$. In $\mathcal{G}(5^\#) = 64242462$, we observe the following initial conditions for $s = 66$:

$$n_{66}(5^\#) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

For the first entry, we don't yet have any occurrences of $s = 66$. We do have two driving terms of length three: 642 and 246; and two driving terms of length four: 4242 and 2424.

$$w_{66}(5^\#) = \frac{1}{\phi_3(5^\#)} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

and

$$w_{66}^\infty = 2.$$

3.2.2. *Constellation $s = 12, 12$.* The constellation $s = 12, 12$ also corresponds to three consecutive primes in arithmetic progression: $p, p + 12, p + 24$. Since $|s| = 24$, to apply the full dynamic system of Theorem 1.1 we use $p_0 = 11$. In $\mathcal{G}(11^\#)$, we calculate the following, for driving terms from length $j_1 = 2$ to $J = 6$:

$$n_{12,12}(11^\#) = \begin{bmatrix} 0 \\ 2 \\ 20 \\ 48 \\ 58 \end{bmatrix} \quad \text{and} \quad w_{12,12}(11^\#) = \frac{1}{\phi_3(11^\#)} \begin{bmatrix} 0 \\ 2 \\ 20 \\ 48 \\ 58 \end{bmatrix}$$

So for $s = 12, 12$ we have

$$w_{12,12}^\infty = \frac{128}{8 \cdot 4 \cdot 2} = 2.$$

It is interesting that the asymptotic relative population of $s = 12, 12$ is the same as for the constellation $s = 66$.

3.2.3. *Constellation $s = 666$.* The constellation $s = 666$ is the smallest constellation corresponding to four consecutive primes in arithmetic progression. Since $|s| = 18$, we can use $p_0 = 7$. In $\mathcal{G}(7^\#)$, we have the following initial conditions for $s = 666$, for driving terms from length $j_1 = 3$ to $J = 5$:

$$n_{666}(7^\#) = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \quad \text{and} \quad w_{666}(7^\#) = \frac{1}{\phi_4(7^\#)} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

For $s = 666$ we have

$$w_{666}^\infty = \frac{6}{3} = 2.$$

It is interesting that we again have the asymptotic relative population of $w_s^\infty = 2$, although here it is relative to constellations of length $j_1 = 3$.

4. POLIGNAC RESULT FOR ARITHMETIC PROGRESSIONS

Here we establish a strong Polignac result on arithmetic progressions. In [8] we established the following analogue of the Polignac conjecture:

Theorem 4.1. *Let $g = 2n$ and let $Q = q_1 \cdots q_m$ be the product of the distinct prime factors of g . Then g arises as a gap in Eratosthenes sieve and asymptotic ratio*

$$w_g^\infty = \prod_{2 < q_i} \frac{q_i - 1}{q_i - 2}.$$

For a gap $g = 2n$, we can now show that not only does the gap occur in Eratosthenes sieve, but that every feasible repetition of g as a constellation $s = g, \dots, g$ occurs in the sieve. We make this precise below.

Above, we calculated the occurrences for a few small examples. We cannot perform the brute force calculation for five primes in arithmetic progression. The minimal constellation we would need to consider is $s = 30, 30, 30, 30$. To apply Theorem 1.1 we would have to use $p_0 = 57$. However, we could obtain asymptotic results via Theorem 2.1 with $p_0 = 5$. The steps we would take in order to apply Theorem 2.1 to this constellation can be generalized to prove Theorem 4.2 below.

Definition. Let g be a gap and let p_k be the largest prime such that $p_k^\# | g$. Let $s = g, \dots, g$ be a repetition of g of length j_1 . Then the constellation s is *feasible* iff $j_1 < p_{k+1} - 1$.

If it survives subsequent stages of the sieve, a repetition s of length j_1 corresponds to $j_1 + 1$ consecutive primes in arithmetic progression.

Theorem 4.2. *Let g be an even number, and let $Q = q_1 \cdots q_m$ be the product of the distinct prime factors of g , with $q_1 < \dots < q_m$. Let s be a feasible repetition of g of length j_1 . Then s occurs in Eratosthenes sieve with asymptotic weight*

$$w_{g, \dots, g}^\infty = \frac{\phi_1(Q)}{\phi_{j_1+1}(Q)}.$$

Proof. We start in the cycle of gaps $\mathcal{G}(Q)$, in which Q may not be a primorial. $\mathcal{G}(Q)$ consists of $\phi(Q)$ gaps that sum to Q . So we can start with any of the $\phi(Q)$ gaps and continue through the cycle g/Q times, and this concatenation is a driving term for g .

We repeat this concatenation of cycles $\mathcal{G}(Q)$, to identify the driving terms for $s = g, \dots, g$, a feasible repetition of the gap g of length j_1 . Starting with any gap in $\mathcal{G}(Q)$, we continue through $\mathcal{G}(Q)$ for $j_1 \cdot g/Q$ complete cycles, and this is a driving term \tilde{s} for s . Since we can start from any gap in $\mathcal{G}(Q)$, we have $\phi(Q)$ driving terms for s .

Now that we have seeded the construction, we move from $\mathcal{G}(Q)$ back into the cycles of gaps for the primorials, $\mathcal{G}(q_m^\#)$.

Case 1. Q is itself a primorial. In this case $Q = q_m^\#$, and the total number of driving terms for s equals $L_1 \cdot n_s(q_m^\#) = \phi(Q)$. From this, we calculate the asymptotic ratio

$$w_s^\infty = \frac{\phi(q_m^\#)}{\phi_{j_1+1}(q_m^\#)} = \frac{\phi_1(Q)}{\phi_{j_1+1}(Q)}.$$

Case 2. Q is not a primorial. Let $\rho_1 < \rho_2 < \dots < \rho_k$ be the primes less than q_m that are not factors of Q . Let $Q_0 = Q$. For $i = 1, \dots, k$, let $Q_i = Q_{i-1} \cdot \rho_i$.

Let \tilde{s} be a driving term for s in $\mathcal{G}(Q_{i-1})$. Under the recursion to create $\mathcal{G}(Q_i)$, we create ρ_i copies of \tilde{s} in step R2. Then in step R3 we close gaps as indicated by the element wise product $\rho_i * \mathcal{G}(Q_{i-1})$.

The driving term \tilde{s} is composed of j_1 driving terms for g .

$$\tilde{s} = \tilde{s}_1 \tilde{s}_2 \dots \tilde{s}_{j_1}$$

in which each $|\tilde{s}_i| = g$.

The $j_1 + 1$ boundary closures will eliminate copies of \tilde{s} from being a driving term for s . All of the intervals in Theorem 2.1 are multiples of g , and these multiples themselves have prime factors entirely divisible by the factors of g . That is, if $p^\#$ is the largest primorial that divides g , then for all $2 \leq j \leq j_1$, by the feasibility of s , the prime factors of j are factors of $p^\#$.

Since $\rho_i \nmid g$, all of the boundary closures for \tilde{s} occur in different copies of \tilde{s} . Thus, of the ρ_i initial copies of \tilde{s} , $j_1 + 1$ are removed as driving terms for s , and the other $\rho_i - j_1 - 1$ copies remain as driving terms of some length but no longer than the length of \tilde{s} .

We don't know the lengths of the driving terms for s in $\mathcal{G}(Q_i)$, but we do know the total population:

$$\begin{aligned} L_1 \cdot n_s(Q_i) &= (\rho_i - j_1 - 1) \cdot L_1 \cdot n_s(Q_{i-1}) \\ &= (\rho_i - j_1 - 1)(\rho_{i-1} - j_1 - 1) \cdots (\rho_1 - j_1 - 1) \cdot L_1 \cdot n_s(Q_0) \\ &= (\rho_i - j_1 - 1)(\rho_{i-1} - j_1 - 1) \cdots (\rho_1 - j_1 - 1) \cdot \phi(Q). \end{aligned}$$

Continuing this construction, we have $Q_k = q_m^\#$, from which

$$L_1 \cdot n_s(Q_k) = (\rho_k - j_1 - 1)(\rho_{k-1} - j_1 - 1) \cdots (\rho_1 - j_1 - 1) \cdot \phi(Q),$$

and the asymptotic ratio is

$$w_s^\infty = \frac{1}{\phi_{j_1+1}(q_m^\#)} \cdot L_1 \cdot n_s(q_m^\#) = \frac{\phi(Q)}{\phi_{j_1+1}(Q)}.$$

□

Thus every feasible repetition of a gap g occurs in Eratosthenes sieve. Compared to other constellations of length j_1 , the repetition of g of length j_1 has asymptotic weight $\phi_1(Q)/\phi_{j_1+1}(Q)$. To the extent that these constellations survive the sieve, they correspond to $j_1 + 1$ consecutive primes in arithmetic progression.

5. CONCLUSION

We have adapted the dynamic model for populations of gaps, as developed in [8], to apply to populations of constellations. These models for constellations all have the same left and right eigenvectors. The matrix of right eigenvectors R is the upper triangular Pascal matrix with alternating signs, and the matrix of left eigenvectors L is the upper triangular Pascal matrix. The eigenvalues depend on the prime and on the length of the constellation, but these otherwise have the same form.

We have calculated the populations for certain specific constellations containing the gap 2. These correspond to pairs of twin primes in various constellations.

We also calculated the populations for a few constellations that are the repetition of a gap g . These repetitions correspond to consecutive primes in arithmetic progression. We are able to establish a strong Polignac result for arithmetic progressions: for any $g = 2n$, every feasible repetition of g occurs as a constellation in Eratosthenes sieve. The asymptotic ratio for the occurrence of the repetition of g of length j_1 depends only on the prime factors of g and the length j_1 .

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